k-colored kernels

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Abstract

We study k-colored kernels in m-colored digraphs. An m-colored digraph D has k-colored kernel if there exists a subset K of its vertices such that

(i) from every vertex $v \notin K$ there exists an at most k-colored directed path from v to a vertex of K and

(ii) for every $u, v \in K$ there does not exist an at most k-colored directed path between them.

In this paper, we prove that for every integer $k \geq 2$ there exists a $(k+1)$-colored digraph D without k-colored kernel and if every directed cycle of an m-colored digraph is monochromatic, then it has a k-colored kernel for every positive integer k. We obtain the following results for some generalizations of tournaments:

(i) m-colored quasi-transitive and 3-quasi-transitive digraphs have a k-colored kernel for every $k \geq 3$ and $k \geq 4$, respectively (we conjecture that every m-colored l-quasi-transitive digraph has a k-colored kernel for every $k \geq l+1$), and

(ii) m-colored locally in-tournament (out-tournament, respectively) digraphs have a k-colored kernel provided that every arc belongs to a directed cycle and every directed cycle is at most k-colored.

Key words: m-colored digraph, k-colored kernel, k-colored absorbent set, k-colored independent set

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1 Introduction

Let j, k and m be positive integers. A digraph D is said to be *m-colored* if the arcs of D are colored with m colors. Given $u, v \in V(D)$, a directed path from u to v of D, denoted by $u \rightsquigarrow v$, is *j-colored* if all its arcs use exactly j colors and it is represented by $u \rightsquigarrow_j v$. When $j = 1$, the directed path is said to be *monochromatic.* A nonempty set $S \subseteq V(D)$ is a *k-colored absorbent set* if for every vertex $u \in V(D) - S$ there exists $v \in S$ such that $u \rightsquigarrow_j v$ with $1 \leq j \leq k$. A nonempty set $S \subseteq V(D)$ is a called a *k-colored independent set* if for every $u, v \in S$ there does not exist $u \rightsquigarrow_j v$ with $1 \leq j \leq k$. Let D be an m-colored digraph. A set $K \subseteq V(D)$ is called a *k-colored kernel* if K is a k-colored absorbent and independent set.

This notion was introduced in [13] and it is a natural generalization to kernels by monochromatic directed paths (the case of 1-colored kernels) defined first in [7]. It is also a generalization of the classic notion of kernels in digraphs which has widely studied in different contexts and has an extensive literature (see [1] for a recent remarkable survey on the topic). In [13], among other questions, the following problem is solved: given an m-colored digraph D, is it true that D has a k-colored kernel if and only if D has a $(k+1)$-colored kernel? The answer is no for both implications. In fact, it can be proved (see Theorems 6.6 and 6.7 of the already mention thesis) that

(i) for every couple of positive integers k and s with $k < s$, there exists a digraph D and an arc coloring of D such that D has a k-colored kernel and D does not have an s-colored kernel and
(ii) for every couple of positive integers k and s with $k < s$, there exists a digraph D and an arc coloring of D such that D has an s-colored kernel and D does not have a k-colored kernel.

The study of 1-colored kernels in m-colored digraphs already has a relatively extense literature and has explored sufficient conditions for the existence of such kernels in many infinite families of special digraphs as tournaments (particularly, in connection with so called Erdős' problem, see for instance [15], [14], [11] and [12]) and its generalizations (multipartite tournaments and quasi-transitive digraphs). As well, it has been of interest searching coloring conditions on subdigraphs of general digraphs to guarantee the existence of 1-colored kernels. These results inspire this work. Taking into account these kind of theorems, it is natural to ask which of them can be generalized or could be weakened in their conditions so we can ensure that there exists a k-colored kernel. In this paper we begin to take the first steps toward this goal and exhibit some results of this type.

In Section 3, we show some results for general digraphs. In particular, we prove that for every integer $k \geq 2$ there exists a $(k+1)$-colored digraph without k-colored kernel and if every directed cycle of an m-colored digraph is monochromatic, then it has a k-colored kernel for every positive integer k. In Section 4, we show that m-colored quasi-transitive and 3-quasi-transitive digraphs have k-colored kernel for every $k \geq 3$ and $k \geq 4$,

respectively. These theorems provide evidences to conjecture that every m-colored l-quasi-transitive digraph (see its definition in the preliminaries) has a k-colored kernel for every $k \geq l+1$. Finally, it is proved that an m-colored locally in-tournament (out-tournament, respectively) digraph has a k-colored kernel if

(i) every arc belongs to a directed cycle and
(ii) every directed cycle is at most k-colored.

2 Preliminaries

In general, we follow the terminology and notations of [3]. If T is directed path of D and $w, z \in V(T)$, then $w \rightsquigarrow^T z$ denotes the directed subpath from w to z along T.

A subdigraph D' of an m-colored digraph D is said to be *monochromatic* if every arc of D' is colored alike and let *colors*(D') denote the set of colors used by the arcs of D'.

Recall that a *kernel* K of D is an independent set of vertices so that for every $u \in V(D) \setminus K$ there exists $(u, v) \in A(D)$, where $v \in K$. We say that a digraph D is *kernel perfect* if every nonempty induced subdigraph of D has a kernel. An arc $(u, v) \in A(D)$ is *asymmetrical* (resp. *symmetrical*) if $(v, u) \notin A(D)$ (resp. $(v, u) \in A(D)$).

Given an m-colored digraph D, we define the *k-colored closure* of D, denoted by $\mathfrak{C}_k(D)$, as the digraph such that $V(\mathfrak{C}_k(D)) = V(D)$ and

$$A(\mathfrak{C}_k(D)) = \{(u, v) : \exists\, u \rightsquigarrow_j v, 1 \leq j \leq k\}.$$

Remark 1 *Observe also that every m-colored digraph D has a k-colored kernel if and only if $\mathfrak{C}_k(D)$ has a kernel.*

We will use the following theorem of P. Duchet [6].

Theorem 1 *If every directed cycle of a digraph D has a symmetrical arc, then D is kernel-perfect.*

In [5] (see Corollary 2 on page 311) it is proved that in a digraph D, there exists a set $B \subseteq V(D)$ such that

(I) no directed path joins two distinct vertices of B and
(A) each vertex $u \notin B$ is the initial endpoint of a directed path finishing in a vertex of B.

Let us define a set of vertices $B \subseteq V(D)$ to be a *kernel by directed paths of* D, if it satisfies conditions (I) (independence by directed paths) and (A) (absorbency by directed paths). Therefore, the result stated before can be settled as

Theorem 2 *Every digraph D has a kernel by directed paths.*

A digraph D is called *k-quasi-transitive* if whenever distinct vertices $u_0, u_1, \ldots, u_k \in V(D)$ such that

$$u_0 \longrightarrow u_1 \longrightarrow \cdots \longrightarrow u_k$$

there exists at least $(u_0, u_k) \in A(D)$ or $(u_k, u_0) \in A(D)$. When $k = 2$ or 3, the digraph D is simply said to be *quasi-transitive* (introduced in [4]) or 3-*quasi-transitive* (see [2] and [9]) respectively. The following is a well-known result proved in [4].

Proposition 3 ([4], Corollary 3.2) *If a quasi-transitive digraph D has a directed path $u \rightsquigarrow v$ but $(u, v) \notin A(D)$, then either $(v, u) \in A(D)$, or there exist vertices $x, y \in V(D) - \{u, v\}$ such that*

$$u \longrightarrow x \longrightarrow y \longrightarrow v \text{ and}$$
$$v \longrightarrow x \longrightarrow y \longrightarrow u$$

are directed paths in D.

The next result is a special case of a more general statement proved in [10] ($d(u, v)$ denotes the distance from u to v for $u, v \in V(D)$).

Proposition 4 ([10], Lemma 4.4) *Let D be a 3-quasi-transitive digraph and $u, v \in V(D)$ such that there exists $u \rightsquigarrow v$. Then*

(i) if $d(u, v) = 3$ or $d(u, v) \geq 5$, then $d(v, u) = 1$ (that is $(v, u) \in A(D)$) and
(ii) if $d(u, v) = 4$, then $d(v, u) \leq 4$.

A *tournament* T on n vertices is an orientation of a complete graph K_n. A digraph D is a *locally in-tournament digraph* (resp. *locally out-tournament digraph*) if for every $u \in V(D)$ the set $N^-(u)$ of in-neighbors of u (resp. $N^+(u)$ of out-neighbors of u) induces a tournament.

3 Results for general digraphs

The research on 1-colored kernels (or kernels by monochromatic directed paths) goes back to the classical result of Sands, Sauer and Woodrow (see [14]) who proved that every 2-colored digraph has a 1-colored kernel. A generalization of this theorem for k-colored kernels when $k \geq 2$ is not longer true as the following result shows.

Theorem 5 *For every $k \geq 2$ there exists a $(k+1)$-colored digraph D without k-colored kernel.*

Proof. Let D be a directed cycle $(u_0, u_1, u_2, \ldots, u_{2k+1}, u_0)$ such that arcs (u_{2i}, u_{2i+1}) and (u_{2i+1}, u_{2i+2}) are colored with color i for every $0 \leq i \leq k$ (the subindices of the vertices are taken modulo $2k+2$). First, notice that if for some $0 \leq m \leq 2k+1$, a vertex

$u_m \in V(D)$ belongs to a k-colored kernel K, then $K = \{u_m\}$. Otherwise, if $u_l \in K$ for some $0 \leq l \leq 2k+1$, $l \neq m$, then there exists either $u_m \rightsquigarrow_j u_l$ or $u_l \rightsquigarrow_j u_m$ with $1 \leq j \leq k+1$ and K is not k-colored independent. Now, we prove that the singleton set K is not even k-colored absorbent. By the symmetry of the arc coloring of D, we can assume without loss of generality that $m = 1$ or $m = 2$. If $m = 1$, then there exists $u_2 \rightsquigarrow_{k+1} u_1$ but there does not exist $u_2 \rightsquigarrow_k u_1$ and K is not k-colored absorbent. Analogously, if $m = 2$, there exists $u_3 \rightsquigarrow_{k+1} u_2$ but there does not exist $u_3 \rightsquigarrow_k u_2$ and K is not k-colored absorbent. ■

Let k be a positive integer. As a direct consequence of Theorem 2, we have the following

Theorem 6 *Let D be an m-colored digraph satisfying that for every $u, v \in V(D)$, if there exists $u \rightsquigarrow v$, then there exists $u \rightsquigarrow_j v$ with $1 \leq j \leq k$. Then D has a k'-colored kernel for every $k' \geq k$.*

Corollary 7 *Let D be an m-colored digraph satisfying that for every $u, v \in V(D)$, if there exists $u \rightsquigarrow v$, then there exists $u \rightsquigarrow v$ of length at most k. Then for every arc coloring of D with m colors ($m \geq 1$), the digraph D has a k-colored kernel for every $k \geq 1$.*

Let us denote by $\mathrm{diam}(D)$ the *diameter* of a digraph D (defined to be $\max\{d(u,v) : u, v \in V(D)\}$ where $d(u,v)$ is the distance from u to v).

Corollary 8 *Let D be an m-colored strongly connected digraph such that $\mathrm{diam}(D) \leq k$. The D has a k'-colored kernel for every $k' \geq k$.*

The following lemma will be an useful tool in proving Theorem 10.

Lemma 9 *Let D be an m-colored digraph such that every directed cycle of D is monochromatic. Let $u, v \in V(D)$ and T_1 denotes the directed path $u \rightsquigarrow v$. If T_2 is a $v \rightsquigarrow u$, then $colors(T_1) = colors(T_2)$.*

Proof. We proceed by induction on the length of T_1, denoted by $l(T_1)$. If $l(T_1) = 1$, then $T_1 \cup T_2$ is a directed cycle which is monochromatic by supposition, so $colors(T_1) = colors(T_2)$. Suppose that the lemma is valid for every $l(T_1) \leq n$ for some positive integer n and we prove the claim for $l(T_1) = n+1$. We have two cases:

Case 1. $V(T_1) \cap V(T_2) = \{u, v\}$. Then $T_1 \cup T_2$ is a directed cycle which is monochromatic and consequently, $colors(T_1) = colors(T_2)$.

Case 2. $|V(T_1) \cap V(T_2)| \geq 3$. Let $z \in V(T_1) \cap V(T_2)$ be the first vertex of T_2 in T_1. Then

$$\gamma = (z \rightsquigarrow^{T_1} v) \cup (v \rightsquigarrow^{T_2} z)$$

is a directed monochromatic cycle. Let T_1' and T_2' be $u \rightsquigarrow^{T_1} z$ and $z \rightsquigarrow^{T_2} u$ respectively. Since $l(T_1') \leq n$, by induction hypothesis, we have that $colors(T_1') = colors(T_2')$. Without loss of generality, suppose that γ is of color 1.

Subcase 2.1. Color 1 does not appear in T_1'. Since $colors(T_1') = colors(T_2')$, then color 1 does not appear in T_2' either and therefore

$$colors(T_1) = colors(T_1') \cup \{1\} = colors(T_2') \cup \{1\} = colors(T_2).$$

Subcase 2.2. Color 1 appears in T_1'. Since $colors(T_1') = colors(T_2')$, then color also appears in T_2' and therefore

$$colors(T_1) = colors(T_1') = colors(T_2') = colors(T_2).$$

■

Theorem 10 *Let D be an m-colored digraph such that every directed cycle of D is monochromatic. Then D has a k-colored kernel for every $k \geq 1$.*

Proof. In virtue of Theorem 1 and Remark 1, we will prove that every directed cycle of $\mathfrak{C}_k(D)$ has a symmetrical arc. So, let $\gamma = (u_0, u_1, \ldots, u_{n-1}, u_0)$ be a directed cycle of $\mathfrak{C}_k(D)$, where n is a positive integer. Then for every $0 \leq i \leq n-1$ there exists $u_i \rightsquigarrow_j u_{i+1}$ in D, denoted by T_i, where $1 \leq j \leq k$ and the indices are taken modulo n. Therefore, $\bigcup_{i=1}^{n-1} T_i$ is a directed walk from u_1 to u_0 and accordingly there exists an $u_1 \rightsquigarrow u_0$ which is denoted by α. Since T_0 is a $u_0 \rightsquigarrow u_1$, by Lemma 9, $colors(T_0) = colors(\alpha)$. Using that T_0 is j-colored with $1 \leq j \leq k$, we conclude that α is a $u_1 \rightsquigarrow_j u_0$ with $1 \leq j \leq k$. Thus, $(u_0, u_1), (u_1, u_0) \in A(\mathfrak{C}_k(D))$ and γ has a symmetrical arc. ■

In case of $k = 1$ in Theorem 10 we have the following

Corollary 11 ([8]) *Let D be an m-colored digraph such that every directed cycle of D is monochromatic. Then D has a kernel by monochromatic directed paths.*

4 k-colored kernels in some generalizations of tournaments

Let T be a tournament. It is well-known that each vertex of maximum out-degree of a tournament T is a king (see Subsection 3.7.1 of [3]). Dually, if x is vertex of maximum in-degree in T, then $y \rightsquigarrow_j x$ with $1 \leq j \leq 2$ for every $y \in V(T)$. Therefore $\{x\}$ is a 2-colored absorbent set and so $\{x\}$ is a 2-colored kernel of T. Observe that a k-colored absorbent set is also l-colored absorbent for every positive integer $l > k$ an we conclude that a tournament T has a k-colored kernel for every $k \geq 2$.

4.1 Quasi-transitive digraphs

A similar result can be proved for quasi-transitive digraphs when $k \geq 3$.

Lemma 12 *Let D be an m-colored quasi-transitive digraph. If there exists $u \rightsquigarrow_j v$ with $1 \leq j \leq k$ and there exists no $v \rightsquigarrow_j u$ with $1 \leq j \leq k$, then $(u,v) \in A(D)$.*

Proof. Consider $u \rightsquigarrow_j v$ with $1 \leq j \leq k$ of minimum length. Using Proposition 3, we have one of the following possibilities:

(i) $(v,u) \in A(D)$ which is impossible by supposition (there exists no $v \rightsquigarrow_j u$ with $1 \leq j \leq k$).

(ii) $(u,v) \in A(D)$ and the conclusion follows.

(iii) There exist vertices $x, y \in V(D) - \{u,v\}$ such that

$$\begin{aligned} &u \longrightarrow x \longrightarrow y \longrightarrow v \text{ and} \\ &v \longrightarrow x \longrightarrow y \longrightarrow u \end{aligned}$$

are directed paths in D. This is also impossible since

$$v \longrightarrow x \longrightarrow y \longrightarrow u$$

is a $v \rightsquigarrow_j u$ with $1 \leq j \leq 3$.

■

Theorem 13 *Let D be an m-colored quasi-transitive digraph. Then D has a k-colored kernel for every $k \geq 3$.*

Proof. We will prove that every directed cycle in $\mathfrak{C}_k(D)$ has a symmetrical arc. By contradiction, suppose that there exists a directed cycle $\gamma = (u_0, u_1, \ldots, u_{n-1}, u_0)$ without symmetrical arcs belonging to $\mathfrak{C}_k(D)$. By the definition of the k-colored closure, there exists $u_i \rightsquigarrow_j u_{i+1}$ with $1 \leq j \leq k$ and there exists no $u_{i+1} \rightsquigarrow_j u_i$ with $1 \leq j \leq k$ in D for every $0 \leq i \leq n-1$ (indices are taken modulo n). By Lemma 12, we have that $(u_i, u_{i+1}) \in A(D)$ for every $0 \leq i \leq n-1$. Therefore γ is a directed cycle of D. On the other hand, γ has at least one change of color in its arcs, otherwise γ would be monochromatic and then $u_{i+1} \rightsquigarrow_j u_i$ with $1 \leq j \leq k$ and for every $0 \leq i \leq n-1$. Without loss of generality, we can assume that the color change occurs at vertex u_1 and so arcs (u_0, u_1) and (u_1, u_2) are colored, say, of colors 1 and 2 respectively. Since D is quasi-transitive, there exists an arc between u_0 and u_2. We consider two cases:

Case 1. $(u_2, u_0) \in A(D)$. Then there exists the directed path $u_1 \longrightarrow u_2 \longrightarrow u_0$ which is at most 2-colored, a contradiction since there is no $u_1 \rightsquigarrow_j u_0$ with $1 \leq j \leq k$ and $k \geq 3$.

Case 2. $(u_0, u_2) \in A(D)$. Since D is quasi-transitive, there exists an arc between u_0 and u_3. If $(u_3, u_0) \in A(D)$, then we have a contradiction similar to Case 1 with vertices u_0, u_2 and u_3. So, $(u_0, u_3) \in A(D)$ and the procedure continues. Since $(u_{n-1}, u_0) \in A(D)$, there exists a first index i such that $1 \leq i \leq n-1$ and $(u_i, u_0) \in A(D)$. Then $(u_0, u_{i-1}) \in A(D)$. But then there exists an at most 2-colored directed path $u_i \longrightarrow u_0 \longrightarrow u_{i-1}$ and this is a contradiction to the fact that there is no $u_i \rightsquigarrow_j u_{i-1}$ with $1 \leq j \leq k$ in D. ■

We remark that an analogous argument as before can be used to prove that an m-colored quasi-transitive digraph D such that every directed triangle is monochromatic has a k-colored kernel for $k = 1, 2$. We do not know if for the case $k = 2$, the monochromacity of the directed triangles can be weakened.

4.2 3-quasi-transitive digraphs

A more elaborated proof allows us to show that a 3-quasi-transitive digraph has a k-colored kernel for every $k \geq 4$.

Let us define the *flower* F_r with r petals as the digraph obtained by replacing every edge of the star $K_{1,r}$ by a symmetrical arc. If every edge of the complete graph K_n is replaced by a symmetrical arc, then the resulting digraph D on n vertices is symmetrical semicomplete.

Remark 2 *Let F_r be an m-colored flower such that $r \geq 1$. Then $\mathfrak{C}_k(F_r)$ with $k \geq 2$ is a symmetrical semicomplete digraph.*

Theorem 14 *Let D be an m-colored* 3*-quasi-transitive digraph. Then D has a k-colored kernel for every $k \geq 4$.*

Proof. Applying Theorem 1 and Remark 1, we will prove that every directed cycle of $\mathfrak{C}_k(D)$ has a symmetrical arc. By contradiction, suppose that $\gamma = (u_0, u_1, \ldots, u_p, u_0)$ is a cycle in $\mathfrak{C}_k(D)$ without any symmetrical arc. Observe that if $p = 1$, then γ has a symmetrical arc and we are done. So, assume that $p \geq 2$.

Using Proposition 4, if there exists an arc $(u_i, u_{i+1}) \in A(\mathfrak{C}_k(D))$ with $0 \leq i \leq p$ (indices are taken modulo $p + 1$) of γ which corresponds to a directed path of length at least 3 in D, then there exists an $u_{i+1} \rightsquigarrow u_i$ of length at most 4. This is a contradiction, γ has a symmetrical arc between u_i and u_{i+1}. So, we assume that every arc of γ corresponds to an arc or a directed path of length 2 in D.

Let δ be the closed directed walk defined by the concatenation of the arcs and the directed paths of length 2 corresponding to the arcs of γ.

First, observe that δ contains a directed path of length at least 3, otherwise δ is isomorphic to F_r with $r \geq 2$ and by Remark 2, γ has a symmetrical arc, a contradiction.

If δ contains a flower F_r, $r \geq 1$ such that two consecutive vertices u_i and u_{i+1} of γ belong to the vertices of F_r, then by Remark 2, there exists a symmetrical arc between u_i and u_{i+1}. Therefore we can assume that

$$\text{there are no consecutive vertices of } \gamma \text{ in a flower.} \tag{*}$$

Observe that if $\delta = \gamma$, as we will see, the same argument of the proof will work even easier.

Let $\delta = (y_0, y_1, ..., y_s)$. Observe that there exist $y_{i_0}, y_{i_1}, ...y_{i_p} \in V(\delta)$ such that $i_j < i_{j+1}$ and $u_l = y_{j_l}$, where $0 \leq l \leq p$.

We define $\varepsilon = (y_i, y_{i+1}, \ldots, y_{i+l})$ of minimum length ($0 \leq i \leq s$ and the indices are taken modulo $s+1$) such that

(i) $y_i = y_{i+l}$, $l \geq 3$,
(ii) $y_i \neq y_t$ for $i+1 \leq t \leq i+l-1$,
(iii) if $y_q = y_r$, then $q = r+2$ ($i+1 \leq q, r \leq i+l-1$),
(iv) there exist $y_{i_1}, y_{i_2}, \ldots, y_{i_{k+1}} \in V(\varepsilon)$ such that $y_{i_1} = u_j$, $y_{i_2} = u_{j+1}$, $\ldots$, $y_{i_{k+1}} = u_{j+k}$ with $k \geq 1$, and
(v) $y_{i+1} \neq y_{i+l-1}$.

Claim 1 *There exists ε subdigraph of δ.*

Since δ is a closed walk, $p \geq 2$ and using (*), condition (i) is satisfied. If there exists $t < l$ such that $y_i = y_{i+t}$, then, by the minimality of ε, $t = 2$ and $l - t = 2$ and therefore $(i+l) - (i+t) = 2$. By (i), we have that $y_i = y_{i+t} = y_{i+l}$ and so $l = 4$. We obtain that

$$y_{i+1} \longleftrightarrow y_i = y_{i+2} = y_{i+4} \longleftrightarrow y_{i+3}.$$

Using that every arc of γ corresponds to an arc or a directed path of length 2, there exist two consecutive vertices of γ in the closed walk

$$y_i \longrightarrow y_{i+1} \longrightarrow y_{i+2} \longrightarrow y_{i+3} \longrightarrow y_{i+4}.$$

Therefore, we have two consecutive vertices of γ in a flower of two petals, a contradiction to the assumption (*). Condition (iii) follows from the minimality of ε and (iv) is immediate from the definition of δ and the fact that $l \geq 3$. If $y_{i+1} = y_{i+l-1}$, then by (iii), $l = 4$ and therefore

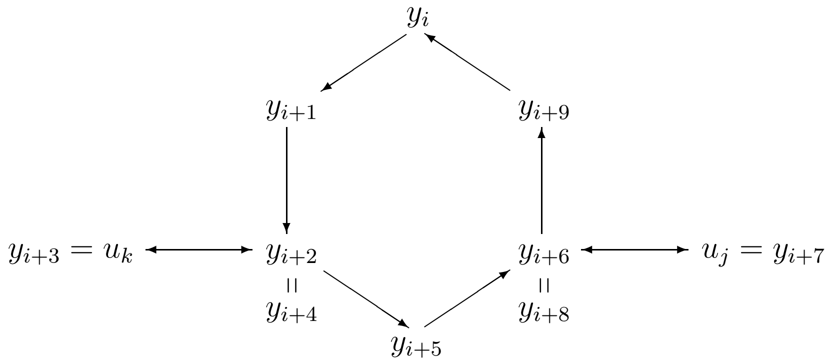
$$y_{i+4} = y_i \longleftrightarrow y_{i+1} = y_{i+3} \longleftrightarrow y_{i+2}$$

which is a flower with two petals and again there are two consecutive vertices of γ in a flower of two petals, a contradiction to the assumption (*). Condition (v) follows.

Claim 1 is proved.

We remark that as a consequence of this claim and since δ is not a flower by supposition, ε is a directed cycle of length at least 3 with perhaps symmetrical arcs attached to some vertices (maybe none) of the cycle for which the exterior endpoints are vertices of γ. For example, closed walk ε with a directed cycle of length 6 and two symmetrical arcs attached is depicted in the next figure. Observe that in this example, y_{i+1}, y_{i+5} and y_{i+9} are elements

of γ by the definition of δ.



Let us rename $\varepsilon = (y_0, y_1, \ldots, y_l)$. By (v) of the definition of ε, we have that $y_1 \neq y_{l-1}$ and by (iv), there exist consecutive $u_0, u_1, \ldots, u_k \in V(\gamma)$ in ε with $k \geq 1$. Notice that u_0 and u_k could not be consecutive vertices of γ and similarly, $(y_{l-1}, y_0) \in A(\varepsilon)$ could not be an arc of γ. Let $u_1 = y_i$ be the second vertex of γ from y_0. Observe that $1 \leq i \leq 3$ by the definition of ε and either $u_0 = y_0$ $(1 \leq i \leq 2)$ or $u_0 = y_1$ $(2 \leq i \leq 3)$.

Claim 2 *$(u_1, y_j) \in A(D)$ for some $l - 2 \leq j \leq l - 1$.*

To prove the claim, let q be the maximum index such that $(u_1, y_q) \in A(D)$ and $q \leq l - 3$. Consider $y_q \longrightarrow y_{q+1} \longrightarrow y_{q+2}$. Observe that by the maximality of q, $y_{q+2} \neq y_q$. Then

$$u_1 \longrightarrow y_q \longrightarrow y_{q+1} \longrightarrow y_{q+2}$$

is a directed path and $q+2 \leq l-1$. Since D is 3-quasi-transitive and using the maximality of q, we have that $(y_{q+2}, u_1) \in A(D)$. There are no consecutive vertices u_i and u_{i+1} of γ in $\{y_q, y_{q+1}, y_{q+2}\}$, otherwise there exists a $u_{i+1} \rightsquigarrow u_i$ of length at most 4 and therefore γ has a symmetrical arc between u_i and u_{i+1}, a contradiction. Thus, the only possibility is that $y_{q+1} = u_t$ for some $2 \leq t \leq k$, $y_{q+3} \in V(\gamma)$ and $y_{q+3} \neq u_t$. There exists the directed path

$$u_1 \longrightarrow y_q \longrightarrow y_{q+1} = u_t \longrightarrow y_{q+2} \longrightarrow y_{q+3}$$

such that $y_{q+3} \neq u_0$, otherwise there exists $u_1 \rightsquigarrow u_0$ of length at most 4 and therefore γ has a symmetrical arc between u_0 and u_1, a contradiction. Therefore $y_{q+3} = u_{t+1}$ with $t+1 \leq k$. Since D is 3-quasi-transitive, there exists an arc between y_q and y_{q+3}. If $(y_{q+3}, y_q) \in A(D)$, then there exists

$$y_{q+3} = u_{t+1} \longrightarrow y_q \longrightarrow y_{q+1} = u_t$$

of length 2 and γ has a symmetrical arc between u_t and u_{t+1}, a contradiction. So $(y_q, y_{q+3}) \in A(D)$. Consider the vertex y_{q+4} (observe that $q + 3 \leq l - 1$). We have that $y_{q+4} \neq y_{q+2}$, otherwise

$$u_{t+1} = y_{q+3} \longrightarrow y_{q+4} = y_{q+2} \longrightarrow u_1 \longrightarrow y_q \longrightarrow y_{q+1} = u_t$$

is a directed path of length 4 and γ has a symmetrical arc between u_t and u_{t+1}, a contradiction. We notice that $y_{q+4} \notin \{u_0, y_0 = y_l\}$, otherwise there exists either the directed

path

$$u_1 \longrightarrow y_q \longrightarrow y_{q+3} = u_{t+1} \longrightarrow y_{q+4} = y_0 = u_0 \text{ or}$$
$$u_1 \longrightarrow y_q \longrightarrow y_{q+3} = u_{t+1} \longrightarrow y_{q+4} = y_0 \longrightarrow y_1 = u_0.$$

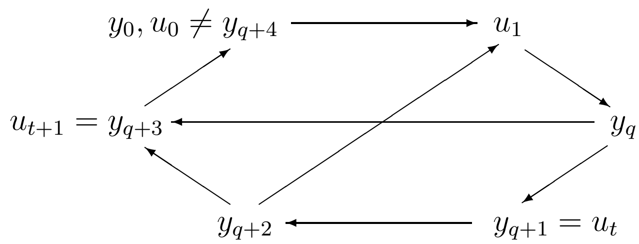
Then, there exists $u_1 \rightsquigarrow u_0$ of length at most 4 and thus γ has a symmetrical arc between u_0 and u_1, a contradiction. Therefore $q+4 \le l-1$ and $y_{q+4} \ne y_q$ by the maximality of q. Since there exists the directed path

$$u_1 \longrightarrow y_q \longrightarrow y_{q+3} = u_{t+1} \longrightarrow y_{q+4},$$

we have that there is an arc between u_1 and y_{q+4} (recall that D is 3-quasi-transitive). By the maximality of q, we obtain that $(y_{q+4}, u_1) \in A(D)$, but in this case we have the directed path

$$u_{t+1} = y_{q+3} \longrightarrow y_{q+4} \longrightarrow u_1 \longrightarrow y_q \longrightarrow y_{q+1} = u_t$$

and there exists $u_{t+1} \rightsquigarrow u_t$ of length 4 and thus γ has a symmetrical arc between u_t and u_{t+1}, a contradiction. The procedure of the claim is illustrated in the following figure.



Claim 2 is proved.

As a consequence of this claim and since either $u_0 = y_0$ or $u_0 = y_1$, we have the directed path $u_1 \longrightarrow y_j \rightsquigarrow u_0$ $(l-2 \le j \le l-1)$ of length at most 4 and γ has a symmetrical arc between u_0 and u_1, a contradiction concluding the proof. ■

The results obtained for quasi-transitive and 3-quasi-transitive digraphs suggest that

Conjecture 15 *Let D be an m-colored l-quasi-transitive digraph. Then D has a k-colored kernel for every $k \ge l+1$.*

4.3 Locally in- and out-tournament digraphs

Lemma 16 *Let D be an m-colored locally out-tournament digraph such that*

(i) every arc belongs to a directed cycle and
(ii) every directed cycle is at most k-colored.

If there exists an $u \rightsquigarrow v$, then there exists a $v \rightsquigarrow_j u$ with $1 \leq j \leq k$.

Proof. Let
$$\beta : u = u_0 \longrightarrow u_1 \longrightarrow \cdots \longrightarrow u_n = v$$
be a directed path from u to v of minimum length. Let $u_j \in V(D)$ be such that $1 \leq j \leq n$ is the maximum index such that u_0 and u_j belong to a same cycle

$$\gamma = (u_0 = w_0, w_1, \ldots, w_{l-1}, w_l = u_j, w_{l+1}, \ldots, w_m = u_0)$$

where $m \geq 1$. If $j = n$, we are done since every cycle of D is at most k-colored. Let us suppose that $j < n$ and consider the vertex $u_{j+1} \in V(\beta)$. Since D is locally out-tournament, $(u_j, u_{j+1}) \in A(D)$ and $(u_j, w_{l+1}) \in A(D)$, there exists an arc between u_{j+1} and w_{l+1}. If $(u_{j+1}, w_{l+1}) \in A(D)$, then there exists a directed cycle

$$(u_0 = w_0, w_1, \ldots, w_{l-1}, w_l = u_j, u_{j+1}, w_{l+1}, \ldots, w_m = u_0)$$

which contradicts the maximality of the choice of j. Therefore there exists $(w_{l+1}, u_{j+1}) \in A(D)$. Since D is locally out-tournament, there exists an arc between u_{j+1} and w_{l+2}. If $(u_{j+1}, w_{l+2}) \in A(D)$, then there exists a directed cycle

$$(u_0 = w_0, w_1, \ldots, w_l = u_j, w_{l+1}, u_{j+1}, w_{l+2}, \ldots, w_m = u_0)$$

which contradicts the maximality of the choice of j. Therefore there exists $(w_{l+2}, u_{j+1}) \in A(D)$. The procedure continues until we obtain that there exists $(w_{m-1}, u_{j+1}) \in A(D)$. Again, since D is locally out-tournament, there exists an arc between u_{j+1} and u_0. If there exists $(u_{j+1}, u_0) \in A(D)$, then there exists a directed cycle

$$(u_0 = w_0, w_1, \ldots, w_l = u_j, u_{j+1}, u_0),$$

a contradiction to the maximality of j. If there exists $(u_0, u_{j+1}) \in A(D)$, then the directed path
$$u = u_0 \longrightarrow u_{j+1} \longrightarrow u_{j+2} \longrightarrow \cdots \longrightarrow u_n = v$$
is of shorter length than β, a contradiction to the choice of β. ■

Lemma 17 *Let D be an m-colored locally in-tournament digraph such that*

(i) every arc belongs to a directed cycle and
(ii) every directed cycle is at most k-colored.

If there exists an $u \rightsquigarrow v$, then there exists a $v \rightsquigarrow_j u$ with $1 \leq j \leq k$.

Proof. Let
$$\beta : u = u_0 \longrightarrow u_1 \longrightarrow \cdots \longrightarrow u_n = v$$
be a directed path from u to v of minimum length. Let $u_j \in V(D)$ be such that $0 \leq j \leq n-1$ is the minimum index such that u_j and u_n belong to a same cycle

$$\gamma = (u_n = w_0, w_1, \ldots, w_{l-1}, w_l = u_j, w_{l+1}, \ldots, w_m = u_n)$$

where $m \geq 1$. If $j = 0$, we are done since every cycle of D is at most k-colored. Let us suppose that $j > 0$ and consider the vertex $u_{j-1} \in V(\beta)$. Since D is locally in-tournament, $(u_{j-1}, u_j) \in A(D)$ and $(w_{l-1}, u_j) \in A(D)$, there exists an arc between u_{j-1} and w_{l-1}. If $(w_{l-1}, u_{j-1}) \in A(D)$, then there exists a directed cycle

$$(u_n = w_0, w_1, \ldots, w_{l-1}, u_{j-1}, w_l = u_j, w_{l+1}, \ldots, w_m = u_n)$$

which contradicts the minimality of the choice of j. Therefore there exists $(u_{j-1}, w_{l-1}) \in A(D)$. Since D is locally in-tournament, there exists an arc between u_{j-1} and w_{l-2}. If $(w_{l-2}, u_{j-1}) \in A(D)$, then there exists a directed cycle

$$(u_n = w_0, w_1, \ldots, w_{l-2}, u_{j-1}, w_{l-1}, w_l = u_j, \ldots, w_m = u_n)$$

which contradicts the minimality of the choice of j. Therefore there exists $(u_{j-1}, w_{l-2}) \in A(D)$. The procedure continues until we obtain that there exists $(u_{j-1}, w_1) \in A(D)$. Again, since D is locally in-tournament, there exists an arc between u_{j-1} and u_n. If there exists $(u_n, u_{j-1}) \in A(D)$, then there exists a directed cycle

$$(u_n = w_0, u_{j-1}, w_l = u_j, w_{l+1}, \ldots, w_m = u_n),$$

a contradiction to the minimality of j. If there exists $(u_{j-1}, u_n) \in A(D)$, then the directed path

$$u_0 \longrightarrow u_1 \longrightarrow \cdots u_{j-1} \longrightarrow u_n = v$$

is of shorter length than β, a contradiction to the choice of β. ■

Theorem 18 *Let D be an m-colored locally out-tournament (in-tournament, respectively) digraph such that*

(i) every arc belongs to a directed cycle and
(ii) every directed cycle is at most k-colored.

Then D has a k-colored kernel.

Proof. By Lemma 16 (resp. Lemma 17), every arc of $\mathfrak{C}_k(D)$ is symmetrical and therefore $\mathfrak{C}_k(D)$ has a kernel in virtue of Theorem 1. By Remark 1, we conclude that D has a k-colored kernel. ■

In view of Theorems 10 and 18, it is natural to ask about the minimum number of colors of a directed cycle in a digraph D such that D has a k-colored kernel. We state the following

Conjecture 19 *Let D be an m colored digraph such that every arc belongs to a directed cycle and every directed cycle is at most k-colored. Then D has a k-colored kernel.*

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